# Uncertainty Relation: From Inequality to Equality *

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The uncertainty area $\delta(p, q) := [\int W(p, q)^2 \,dp\,dq]^{-1}$ is proposed in place of $\delta p \cdot \delta q$, and it is shown that each pure quantum state is a minimum uncertainty state in this sense: $\delta(p, q) = 2\pi\hbar$. For mixed states, on the other hand, $\delta(p, q) > 2\pi\hbar$. In a phase space of $2F(=6N)$ dimensions, $S := k_B \cdot \log[\delta^F(p, q)/(2\pi\hbar)^F]$ with $\delta^F(p, q) := [\int W(p, q)^2 \,d^F p\, d^F q]^{-1}$ is considered as an alternative to von Neumann's entropy $\tilde{S} := k_B \cdot \mathrm{trc}[\hat{\varrho} \log(\hat{\varrho}^{-1})]$.

## 1. Introduction

When Heisenberg first proposed his uncertainty relations [1] he had in mind an approximate equality rather than an inequality [2]. What he has written is

$$\Delta p \cdot \Delta q \approx h, \qquad \Delta E \cdot \Delta t \approx h \tag{1}$$

with $p := p_x$, $q := q_x$. But what is the (rough) meaning of $\Delta x$ in all these cases of the variable $x$? What Heisenberg envisioned was the lenght $\Delta x := x'' - x'$ of an interval $[x' \mid x'']$ such that $x \in [x' \mid x'']$ in a well qualified majority of cases (with a chance of about 80%, say). The product of such *indeterminacies* for a pair of canonically conjugate variables has to be at least nearly equal to Planck's *quantum of action* $h$ in each natural state of affairs. This latter proviso stipulates that the state considered is not clouded by *additional uncertainties* of a purely subjective nature.

Later authors were not satisfied with this somewhat vague formulation, or with Heisenberg's inductive argument. Very soon the text books rendered the well known inequality

$$\sigma(p) \cdot \sigma(q) \geq \tfrac{1}{2}\hbar \tag{2}$$

where $\hbar = h/2\pi$ is the natural *unit of action*. Here

$$\sigma(x) := \sqrt{v(x)} \tag{3}$$

is the *standard deviation* (or 'dispersion') of the *observable* $\hat{x}$, where $v(x) := \mu((x - \mu(x))^2)$ is the *variance* and $\mu(x)$ the *mean* of $\hat{x}$. The inequality (2) may be readily deduced from the commutation relation $i[p, q] = \hbar$. Unfortunately, a similar statement for the pair $(E, t)$ is not so easily obtained, $t$ being a $c$-number so that $i[E, t] = 0$.

In addition, from Heisenberg's original point of view the quantity of (2) is often *too small* by a factor of at least 10, as can be seen from the Table and Figures 1–3. This is easily taken care of [3] by using the *spread*

$$\delta x := \sqrt{4\pi}\, \sigma(x) \tag{4}$$

instead of the dispersion $\sigma(x)$. Yet in some (not so rare) cases the left hand side of the inequality (2) is much *too large*, sometimes even by an infinite factor.

Table 1. Three uncertainty measures for various distributions.

| Distribution | $p(x)$ | $\sigma(x)$ | $\delta x$ | $\delta[x]$ |
|---|---|---|---|---|
| Rectangular | $r_a(x) = \frac{1}{2} a^{-1} \Theta \cdot (\frac{1}{4} a^2 - x^2)$ | $\frac{a}{\sqrt{12}}$ | $\sqrt{\frac{\pi}{3}}\, a$ | $a$ |
| Lorentzian | $l_a(x) := \frac{a}{2\pi} \cdot (\frac{1}{4} a^2 + x^2)^{-1}$ | $\infty$ | $\infty$ | $2\pi a$ |
| Gaussian | $g_a(x) := \sqrt{2}\, a^{-1} e^{-2\pi x^2/a^2}$ | $\frac{a}{\sqrt{4\pi}}$ | $a$ | $a$ |
| Widely split gaussian | $s_{A,a}(x) := \frac{1}{2} g_a(x+A) + \frac{1}{2} g_a(x-A),\ a \ll A$ | $\approx A$ | $\approx \sqrt{4\pi} A$ | $\approx 2a$ |

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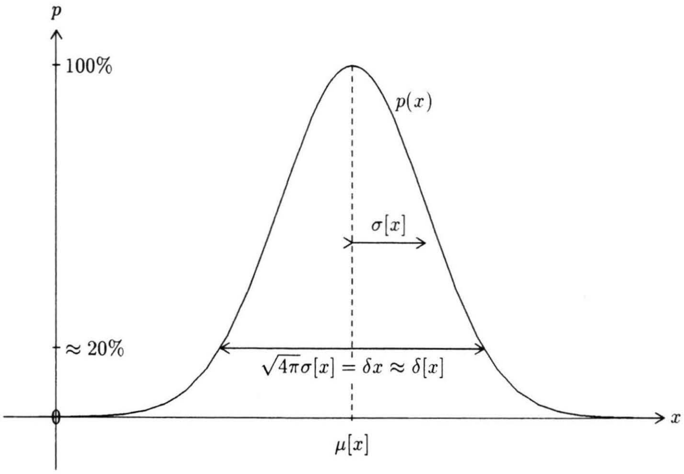


Fig. 1. The width measures $\sigma(x)$, $\delta x$ and $\delta[x]$ of a typical distribution.

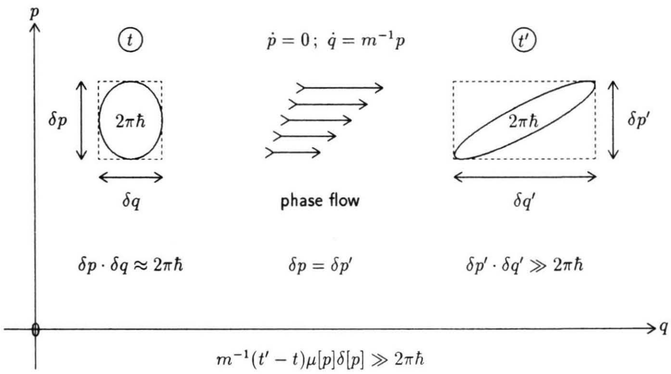


Fig. 2. A freely moving particle sequeezes its state spontaneously, whereby the true uncertainty area is conserved.

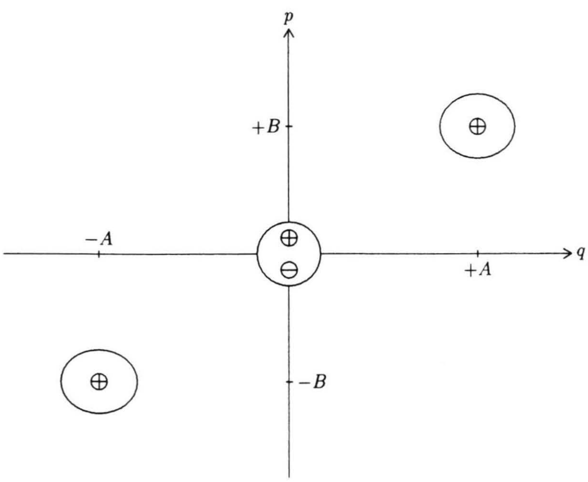


Fig. 3. Wigner's quasi-distribution $W(p, q)$ for a state $\psi(x) := [\phi(x + A) + \phi(x - A)]/\sqrt{2} + \ldots$ consisting of two amply separated branches.

Both of these difficulties can be remedied by the following introduction of an *uncertainty length*:

$$\delta[x] := \frac{1}{\int p(x)^2 \, dx}, \tag{5}$$

where $p(x)$ denotes the probability density[1] of the stochastic variable $\hat{x}$. Thus $\delta[x]$ may be interpreted as the (total) length of (all the) interval(s) that produce rectangle(s) of (total) area $h_x \cdot \delta[x] = 1$ (= 100%), the height $h_x$ being the mean value of the normalized $p(x)$ with $p(x)$ itself as its own weight factor.

[1] The nominator 1 may be replaced by the square of the normalization integral $\int p(x) \, dx$, thus yielding a formulation which is formally emancipated from the normalization condition $\int p(x) \, dx = 1$.

## 2. Uncertainty Area

The inequality (2) suffers from inadequacies of another kind in situations where $p$ and $q$ are highly *correlated*. Typical examples are: (a) a particle after a long free motion [4]; (b) a strongly squeezed state of an oscillator [5]. Under such circumstances the two-dimensional phase space that is effectively claimed will resemble a strongly slanted parallelogram rather than a rectangle. Then, the product $\delta p \cdot \delta q$ will be much larger than some more accurate measure $\delta[p, q]$ of the genuinely inhabited portion of the phase plane.

Can this last idea be made precise? I think *yes*, having in mind a two-dimensional analog of the one-dimensional picture underlying (5). As $p$ and $q$ do not commute, I must resort to Wigner's quasi-distribution $W(p, q)$. This then yields the notion of the *uncertainty area*

$$\delta[p, q] := \frac{1}{\int W(p, q)^2 \,\mathrm{d}p\,\mathrm{d}q}, \tag{6}$$

where the dominator surely is a positive quantity, Wigner's $W$ being real [2]. Physical situations where $W$ attains negative values are typically those with pronounced quantum illocalities, like a particle some time after passing through a beam splitter, as shown in Figure 3. The well-defined quantity $\delta[p, q]$ may serve as an illuminating concept, especially in those extreme cases where the usual $\delta p \cdot \delta q$ gives much too large of an estimate.

Such a misfortune can never happen here, because the *uncertainty equation*

$$\delta[p, q] = 2\pi\hbar \tag{7}$$

turns out to hold true for each and every *pure state* $\hat{\varrho} = |\psi\rangle\langle\psi|$. Here, it is $\varrho(q'', q') := \psi(q'')\psi(q')^*$ which enters

$$W(p, q) := \frac{1}{2\pi\hbar}\int \varrho\left(q + \frac{1}{2}\check{q}, q - \frac{1}{2}\check{q}\right) e^{+ip\check{q}/\hbar}\,\mathrm{d}\check{q}. \tag{8}$$

A physical motivation for this definition [7] including the factors $(2\pi\hbar)^{-1}$ and $\pm\frac{1}{2}$ is given in [8]. The easiest way to prove (7) is by using the identity [7]

$$\int x(p, q)\, W(p, q)\,\mathrm{d}p\,\mathrm{d}q = \mathrm{trc}(\hat{x}\,\varrho). \tag{9}$$

We need only to substitute $2\pi\hbar\, W(p, q)$ for $x(p, q)$ and, correspondingly, $\hat{\varrho} = |\psi\rangle\langle\psi|$ for $\hat{x}$, thus obtaining $\delta[p, q] = 2\pi\hbar/\mathrm{trc}\,\hat{\varrho}^2 = 2\pi\hbar/\mathrm{trc}\,\hat{\varrho} = 2\pi\hbar$.

For an arbitrary state, not necessarily a pure one, we have more generally $O \le \hat{\varrho}^2 \le \hat{\varrho} = \hat{\varrho}^* \le I$, hence $\delta[p, q] \ge 2\pi\hbar$.

## 3. Entropy

This view is supported by a comparison with related concepts of *quantum statistical thermodynamics*. To this end if we generalize from one to $F$ degrees of freedom (where $F = 3N$), and from pure states to mixed ones. Then with $x := (x_1, \ldots, x_F)$ for $x \in \{p, q\}$ we have

$$\delta^F[p, q] \ge (2\pi\hbar)^F. \tag{10}$$

This squares well with the familiar thermodynamical fact that each microstate occupies the phase volume $(2\pi\hbar)^F$ in the mean. Thus we may consider

$$G := \delta^F[p, q]/h^F \quad \text{with} \quad \delta^F[p, q] := h^F/\mathrm{trc}\,\hat{\varrho}^2 \tag{11}$$

as the *mean statistical weight* of the thermodynamical state $\hat{\varrho}$ under consideration. For macroscopic systems, even rather small ones, $G$ has an exorbitant magnitude of about $10^{10^{10}}$, or more. On the other hand, Nernst's Third Law of Thermodynamics states that at zero temperature the limiting value $S = 0$ obtains, corresponding to $G = 1$, which means that the ground state is essentially nondegenerate [3].

The natural logarithm ($\log := \log_e$) of the phase volume $\delta^F[p, q]$, measured in its natural units $(2\pi\hbar)^F$, is a statistical counterpart of the *entropy* notion [9] in units of Boltzmann's constant $k_B$. Thus we may propose

$$S = k_B \cdot \log G \tag{12}$$

$= \mathrm{bit} \cdot \log_2 G \ge 0$ as an alternative closely related to the well-known definition $\tilde{S} := -k_B\,\mathrm{trc}(\hat{\varrho}\log\hat{\varrho})$, originally proposed by Boltzmann, Gibbs, and von Neumann (and later generalized into informatics by Szilard and Shannon). That $S \approx \tilde{S}$ is true for typical macroscopic states can be checked by the prototypical case of a simple yes–no distribution: $G$ microscopic states of equal weight have

$$\begin{aligned} S/k_B &= -\log \Sigma_{\nu=1}^{G} p^2 = -\log(GG^{-2}) \\ &= \log G = -GG^{-1}\log(G^{-1}) \\ &= -\Sigma_{\nu=1}^{G} p \log p = \tilde{S}/k_B. \end{aligned}$$

[2] Again the numerator 1 may be viewed as the square of the normalization integral $\int W(p, q)\,\mathrm{d}p\,\mathrm{d}q = 1$.

[3] It has at most a microscopic multiplicity $G \ll F!$ where $F \gtrsim 10^{12}$.

The microcanonical ensemble, from which Gibbs has derived the canonical one, is a case in point.

## 4. Conclusion

Returning to the phase plane $(p, q)$, we may summarize by stating that *each pure quantum state is a minimum uncertainty state* according to a rather natural definition $\delta[p, q]$ of the combined $(p, q)$-uncertainty. Mixed states have larger uncertainties, their statistical weight $G := \delta[p, q]/2\pi\hbar$ being an appropriate measure of impurity.

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In discussions with Wolfgang Schleich [10] I have realized that my former considerations about $\delta p$, $\delta q$, $\delta[p, q]$ etc. might be useful enough to be published. This fine workshop is a good occasion. It is a pleasure to thank the quantum optics colleagues around Garching and Albuquerque, especially Marlan Scully, for all the hospitality and the agreeable atmosphere of scientific exchange. I owe Berthold-Georg Englert my thanks for advising me on the English text.

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